THE MATHEMATICS OF COORDINATED CONTROL OF PROSTHESES AND MANIPULATORS

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Summary

The problem of coordinated rate control and position control of multidegree of freedom arms are treated together in this paper. Previous work by the author is summarized and revised, and a coherent theory is presented which allows:

1) *real time computer control*
2) *rate control commands expressed in a wide variety of external coordinate systems including hand-oriented coordinates, rectangular or shperical coordinates, or motion along special axes such as line of sight*
3) *any number of command axes to be activated simultaneously*
4) *solution of the position control problem by means consistent with the rate control problem, allowing desired final position to be specified in terms of meaningful external coordinates, and obviating the need for numerical coordinates, and obviating the need for numerical search or solution of complicated equations to find the final joint angles.*
5) *consistent treatment of redundant arms*
6) *attention to singularities*
7) *laison to optimal control of dynamic models of arms*

Introduction

This report briefly summarizes the results of several years' work by the author and his colleagues on the development of coordinated motion control of computer-driven arms. The basic ideas are set out in reference /1/, a first attempt at hardware realization is described in /2/, and the latest and most successful realization is contained in /3/, which also contains details of hardware and the contributions of many co-workers.

The objective of coordinated control is to allow the operator of a mechanical arm to command rates of the arm's hand along coordinate axes which are convenient, task-related, and visible to the operator. A useful set of coordinates, fixed to the hand itself, is shown in Figure 1. To accomplish such motions, several joints of the arm must move simultaneously at time-varying rates. This is extremely difficult to accomplish if conventional rate control (switches connected one-to-one to the joint motors) is used. Some means of coordinating the joint motions is needed,

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to resolve the useful command directions into the necessary joint motions. For this reason, the method described here is called resolved motion rate control. Previous work in this area by others is contained in /4/ - /10/.

The method allows commands to be exerted in a wide variety of coordinate systems in addition to that shown in Figure 1, using arms of any sufficient number of joints. Generally the minimum number of joints equals the number of command directions, but arms with extra joints can be accommodated. The commands in Figure 1 can be called for independently, or superposed in any proportions. For example, reach will occur without the hand's orientation in space changing, since reorientation is controlled by other commands.

Using the commands of Figure 1 as a base, we could mechanize spherical coordinates with arbitrary center, cartesian coordinates, motion along or about axes peculiar to some tool being grasped by the hand, and so on.

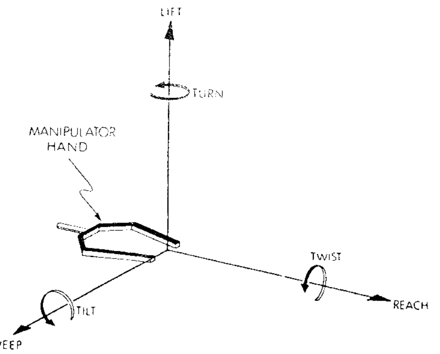


Fig. 1. Hand with hand-oriented coordinate system

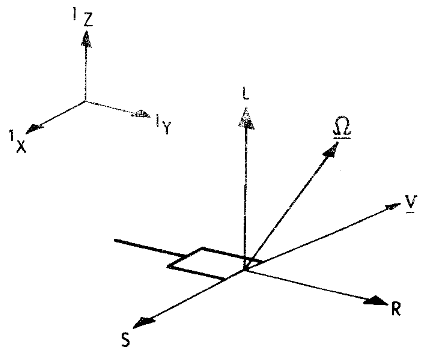


Fig. 2. Hand coordinates related to $\underline{V}$ and $\underline{\Omega}$

Computation of Resolved Rate for Hand-Oriented Commands

Figure 2 shows a manipulator hand with its attached coordinate frame plus $\underline{V}$, the velocity vector of that frame's origin, and $\underline{\Omega}$, the rotation rate vector about that origin. The frame ^1X, ^1Y, ^1Z represents the shoulder or base. The components of $\underline{V}$ along R, L and S give the reach velocity, lift velocity and sweep velocity respectively while $\underline{\Omega}$'s components give the rotation rates about these axes. A six element vector $\dot{\underline{S}}$ representing the command rates along hand axes may be written

$$\dot{\underline{S}} = \begin{bmatrix} \underline{V} \\ - - - \\ \underline{\Omega} \end{bmatrix} \qquad (1)$$

Each component of $\dot{\underline{S}}$ may be expressed as a sum of coefficients times joint angle rates. Call the six joint angles $\underline{\theta}$ and the six joint angle rates $\dot{\underline{\theta}}$. Then $\dot{\underline{S}}$ and $\dot{\underline{\theta}}$ are related by

$$\dot{\underline{S}} = \underset{\sim}{J} \, (\underline{\theta}) \cdot \dot{\underline{\theta}} \qquad (2)$$

where each element in the six by six matrix $J(\underline{\theta})$ depends on $\underline{\theta}$ and is given by

$\underset{\sim}{J}_{ij}$ = i^{th} component of $\dot{\underline{S}}$ per unit $\dot{\theta}_j$ when all other $\dot{\theta}_k = 0$ for $j \neq k$ (3a)

An equivalent expression for this is

$\underset{\sim}{J}_{ij}$ = partial derivative of the i^{th} positional or angular coordinate of the hand with respect to the j^{th} joint angle (3b)

Having obtained $\underset{\sim}{J}$, we may find the required $\dot{\underline{\theta}}$ by inverting $\underset{\sim}{J}$ to

$$\dot{\underline{\theta}} = \underset{\sim}{J}^{-1} \dot{\underline{S}} \qquad (4)$$

For example, if the user wants the hand to lift, (4) will generate the required $\dot{\underline{\theta}}$. Since the commanded rotation rates are zero, the hand will not rotate while reaching but will keep a fixed orientation in space.

Calculation of J by Vector Cross Products

The vector cross product method /11/ for computing J is indicated in Figure 3. A coordinate frame is assumed to be attached to each joint, the j^{th} frame at the j^{th} joint, $j = 1, \ldots, 6$, with the hand frame being the 7^{th} and frame 1 at the shoulder. O_j is the origin of frame j. In Figure 3, the j^{th} joint, the shoulder frame and the hand are shown, together with the unit vector $\underline{u}_j$ along the axis of $\dot{\theta}_j$, the vector $\underline{b}_{j7}$ from O_j to O_7, and $\underline{V}_j$ and $\underline{\Omega}_j$ which result from $\dot{\theta}_j$ if all other $\dot{\theta}$'s are zero. Then

$$\underline{V}_j = \underline{u}_j \times \underline{b}_{j7} \cdot \dot{\theta}_j \qquad (5)$$

$$\underline{\Omega}_j = \underline{u}_j \cdot \dot{\theta}_j \qquad (6)$$

where all vectors are expressed in frame 1. Using vectors $\underline{V}_j$ and $\underline{\Omega}_j$ as is will allow us to obtain hand motion along or around frame 1 axes, useful, for example, for generating lift along a fixed vertical axis. To obatin motion along or around frame 7 axes, we need to express $\underline{V}_j$ and $\underline{\Omega}_j$ in hand coordinates. This is done by multiplying by a 3 x 3 rotation matrix. The upper left 3 x 3 partition of $\underset{\sim}{AEQ}_7$* is called ${}^1\underset{\sim}{C}_7$ because it expresses frame 7 vectors in frame 1. Its transpose ${}^1\underset{\sim}{C}_7^T$ is ${}^7\underset{\sim}{C}_1$, which expresses frame 1 vectors in frame 7. Thus

$$ {}^1\underset{\sim}{C}_7^T \cdot \begin{bmatrix} \underline{V}_j \\ --- \\ \underline{\Omega}_j \end{bmatrix}, \quad j = 1, \ldots, 6 \qquad (7) $$

expresses $\underline{V}_j$ and $\underline{\Omega}_j$ in hand coordintes. Thus the ${}^7Y$ component of $\underline{V}_j$ is $\dot{\theta}_j$'s contribution to reach, and the ${}^7Y$ component of $\underline{\Omega}_j$ is $\dot{\theta}_j$'s contribution to reach, and the ${}^7Y$ component of $\underline{\Omega}_j$ is $\dot{\theta}_j$'s contribution to twist. The column vector in (7), when divided by $\dot{\theta}_j$, gives the j^{th} column of $\underset{\sim}{J}$ according to (3). Thus

$$ J = {}^1\underset{\sim}{C}_7^T \left[\begin{array}{c|c|c|c} \underline{u}_1 \times \underline{b}_{17} & \underline{u}_2 \times \underline{b}_{27} & \text{etc.} & \underline{u}_6 \times \underline{b}_{67} \\ ---- & ---- & & ---- \\ \underline{u}_1 & \underline{u}_2 & & \underline{u}_6 \end{array} \right] \qquad (8) $$

Although the Argonne E-2 has only turn joints, some manipulators have sliding joints. If the j^{th} joint is a slider then θ_j is fixed and s_j, the slide coordinate, is a variable. The j^{th} column of $\underset{\sim}{J}$ is then

$$ {}^1\underset{\sim}{C}_7^T \begin{bmatrix} \underline{V}_j \\ --- \\ \underline{0} \end{bmatrix} / \dot{s}_j, \quad j = 1, \ldots \quad 6 \qquad (9) $$

where $\underline{V}_j = \underline{u}_j \cdot \dot{s}_j$

and the lower 3 x 1 partition is zero. $\underline{u}_j$ is, as before, the unit vector along ${}^j\underline{Z}$, the slide direction as shown in Figure 3.

Other Coordinate Directions

As an example of other possible coordinate directions which

*$\underset{\sim}{AEQ}_7$ is a 4 x 4 transformation matrix which converts the position and orientation of frame 7 into the position and orientation of frame 1. See /3/ for details.

can be mechanized, consider reach motion along a line of sight from an eye to the hand. Let the eye be at O_8, the origin of frame 8. Then the desired reach motion is to be along the vector $\underline{b}_{87}$ and will be obtained if the second row of $\underset{\sim}{J}$ in (8) is replaced by

$$V_j' = (\underline{V}_j^T \, \hat{\underline{b}}_{87})/\dot{\theta}_j \qquad \text{for } j = 1, \ldots, 6 \qquad (10)$$

where conversion by ${}^1\underset{\sim}{C}_7^T$ is omitted, and

$$\underline{V}_j = \underline{u}_j \times \underline{b}_{j7} \cdot \dot{\theta}_j \quad \text{as before}$$

$\hat{\underline{b}}_{87}$ = unit vector along $\underline{b}_{87}$ expressed in frame 1 coordinates

$(\underline{a}^T\underline{b})$ = dot product of $\underline{a}$ and $\underline{b}$

Methods for Computing $\underset{\sim}{J}$

We have explored two methods for obtaining $\underset{\sim}{J}$ in real time. Each has some advantages. The first method used was numerical interpolation. For this $\underset{\sim}{J}^{-1}$ was calculated at a number of joint angle values, corresponding to a center position, a positive extension and a negative extension for each joint, the other joints being centered when each joint was extended. This designated 13 points at which $\underset{\sim}{J}^{-1}$ was precalculated and stored. Values of $\underset{\sim}{J}^{-1}$ at arbitrary points were computed by interpolation with good accuracy. The value of this method lies in its use of read-only memory and fairly rapid computing time. Its disadvantage is that accuarcy is preserved only within a region of Θ values somewhat smaller than the arm's useful range.

The other procedure is to calculate $\underset{\sim}{J}$ using (8) in real time. This uses eraseable storage and takes slightly longer than the

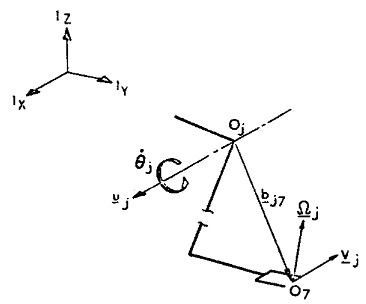


Fig. 3. Illustration of vector cross-product method for computing J

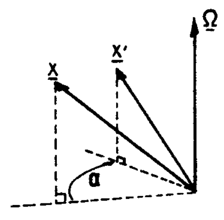


Fig. 4. Definition of angle α

interpolation, but it allows almost unlimited motion of the hand. It is the method currently in use.

Extension to Position Control

For position to position control we assume that the hand is in some position and orientation such that $\underline{b}_{17}(\underline{\theta}_i)$ and ${}^1\underset{\sim}{C}_7(\underline{\theta}_i)$ are known, and that we desire the hand to move to a new position whose $\underline{b}_{17}(\underline{\theta}_f)$ and ${}^1\underset{\sim}{C}_7(\underline{\theta}_f)$ are given. $\underline{\theta}_f$ itself, the new joint angles, is not assumed to be known. Information in the form $\underline{b}_{17}(\underline{\theta}_f)$ and ${}^1\underset{\sim}{C}_7(\underline{\theta}_f)$ could come from a pointing system or other information source describing the location and grasp direction of some object we want grasped. Assuming that the hand is to move to the new position in a time interval T, we have

$$\underline{V} = {}^1\underset{\sim}{C}_7^T(\underline{\theta})\,[\underline{b}_{17}(\underline{\theta}_f) - \underline{b}_{17}(\underline{\theta}_i)]/T \tag{11}$$

= first three components of $\dot{\underline{S}}$ for use in eq. (4)

This says that $\underline{V}$ in hand coordinates sweep, reach, and lift is obtained from the initial vector difference in $\underline{b}_{17}$ (in shoulder coordinates) projected into current hand coordinates by ${}^1\underset{\sim}{C}_7^T(\underline{\theta})$. The current $\underline{\theta}$ must be obtained either by reading the joint angles or computing the integral

$$\underline{\theta} = \underline{\theta}_i + \int_0^t \dot{\theta}\,dt \tag{12}$$

The last three components of $\dot{\underline{S}}$ are obtained by finding an axis vector $\underline{\Omega}$ about which frame 7 should turn so as to change ${}^1\underset{\sim}{C}_7(\underline{\theta}_i)$ into ${}^1\underset{\sim}{C}_7(\underline{\theta}_f)$. A matrix C_{if} exists which will accomplish this rotation. These three matrices are related by

$${}^1\underset{\sim}{C}_7(\underline{\theta}_f) = {}^1C_7(\underline{\theta}_i)C_{if} \tag{13}$$

so that

$$\underset{\sim}{C}_{if} = {}^1\underset{\sim}{C}_7^T(\underline{\theta}_i)\,{}^1\underset{\sim}{C}_7(\underline{\theta}_f) \tag{14}$$

The desired rotation axis $\underline{\Omega}$ is the one vector in hand coordinates which is unchanged during this rotation. That is

$$\underset{\sim}{C}_{if}\underline{\Omega} = \underline{\Omega} \tag{15}$$

This means that $\underline{\Omega}$ is the eigenvector of $\underset{\sim}{C}_{if}$ with unit eigenvalue. The angle α through which frame 7 turns about axis $\underline{\Omega}$ may be obtained from examination of Figure 4. Here $\underline{\Omega}$ is the axis vector and is

assumed to have been normalized to unit length. $\underline{X}$ and $\underline{X}'$ are the original and final unit vectors along the X axis of frame 7, described here for convenience in original frame 7 coordinates. α is measured in the plane normal to $\underline{\Omega}$. The X component of $\underline{\Omega}$ is Ω_x and by defini- of $\underline{\Omega}$ this is the same as the $\underline{X}'$ component of $\underline{\Omega}$, $\Omega_{x'}$.
That is

$$\Omega_x = (\underline{\Omega}^T\underline{X}) = (\underline{\Omega}^T\underline{X}') = \Omega_{x'} = \text{first element of } \underline{\Omega} \qquad (16)$$

The projection of $\underline{X}$ onto the plane normal to $\underline{\Omega}$ is $\underline{X} - \Omega_x\underline{\Omega}$ and the coresponding projection of $\underline{X}'$ is $\underline{X}' - \Omega_x\underline{\Omega}$. The dot product of unit vectors along these projections gives α:

$$\cos\alpha = \frac{(\underline{X} - \Omega_x\underline{\Omega})^T(\underline{X}' - \Omega_x\underline{\Omega})}{|\underline{X} - \Omega_x\underline{\Omega}| \cdot |\underline{X}' - \Omega_x\underline{\Omega}|} \qquad (17)$$

$$= \frac{(\underline{X}^T\underline{X}') - \Omega_x^2}{1 - \Omega_x^2} \qquad (18)$$

(In case $\underline{\Omega}$ lies along $\underline{X}$, this formula is inapplicable. Replace $\underline{X}$ with $\underline{Y}$ and $\underline{X}'$ with $\underline{Y}'$ in that case.)

Now that α is known, we need only scale $\underline{\Omega}$ by α/T and then take this as the rotational rate vector for frame 7. It is already expressed in frame 7, so its elements are directly the rates for tilt, turn, and twist, thus providing the last three elements of $\dot{\underline{S}}$. Barring numerical or servo errors, this $\dot{\underline{S}}$ will carry the hand to the new position and orientation. The calculation may be made closed loop (hence less error prone) by defining a new $\underline{\Theta}_i$ periodically along the trajectory, calculating a new $\underline{b}_{17}(\underline{\Theta}_i)$ and a new T for use in eq. (11), plus a new $\underline{C}_{if}$, $\underline{\Omega}$ and α.

This procedure is an improvement over numerical search methods /12/ or direct analytical attacks on the geometric equations /13/. The former can have convergence difficulties while the latter are applicable only to certain types of arm configurations.

Dynamic Control of Arms

The previous sections have derived joint angle rate histories based on input commands and coordination constrains which are purely kinematic. In this section we discuss methods by which arms having inertia may be commanded to follow such trajectories. Previous work in this area is limited to a few papers, notably those of Kahn and Roth /14/, which considers the minimum time control problem for a three-joint arm, and Monster /15/, who studied stabilization and trajectory tracking problems in the Case Arm-Aid. The main roadblock to the study of dynamic control of arms is the sheer difficulty of obtaining the equations of motion. Progress made by Sturges /16/ in computer-generation and, more important, computer-simplification of these equations has made possible the work described below. Computer equation generation requires great amounts of computer memory at present, so that arms with more than four joints have not yet been treated.

It is for this reason that a simple linearization method, applicable to arms with any number of joints, is valuable /17/. The equations of motion are obtained easily and quickly in numerical form by considering a linearization of Lagrange's equations. The result is the following state variable representation:

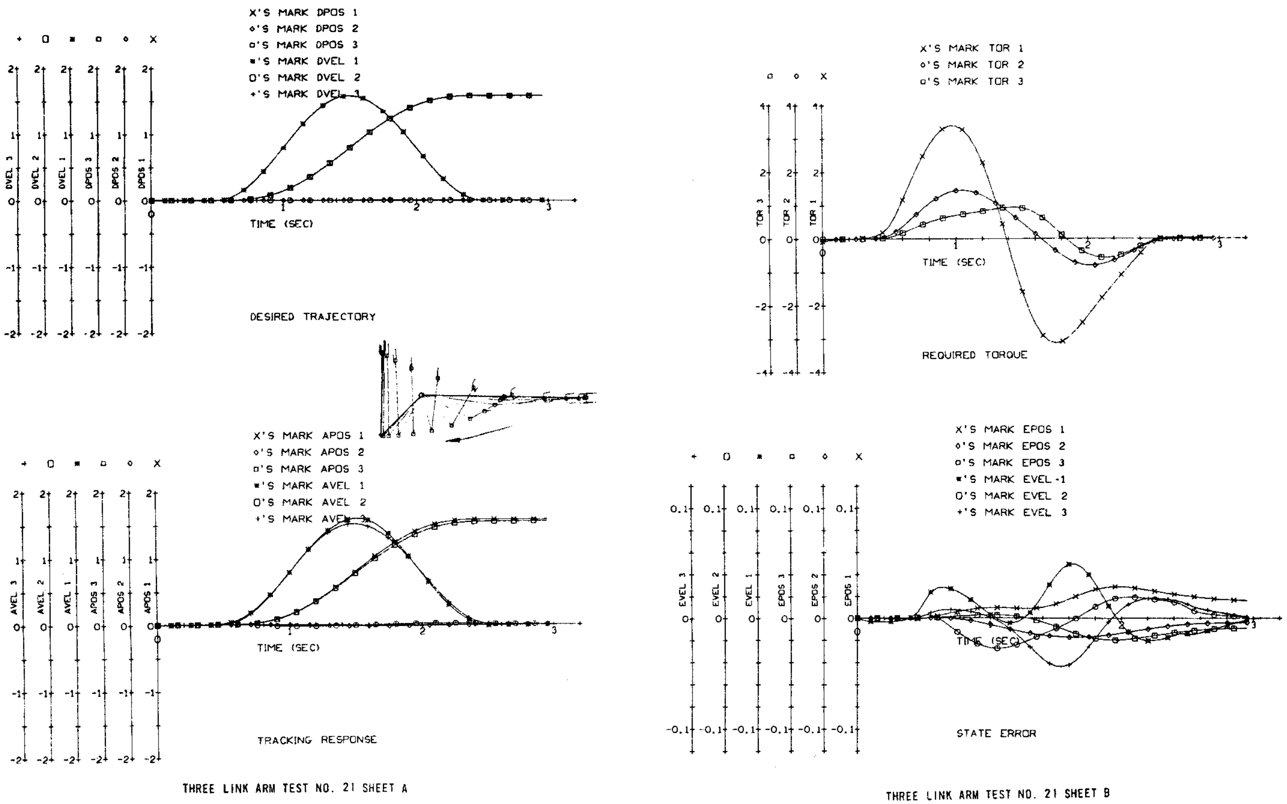
$$\dot{\underline{s}} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{I} \\ -\underset{\sim}{M}^{-1}\underset{\sim}{K} & -\underset{\sim}{M}^{-1}\underset{\sim}{B} \end{bmatrix} \underline{s} + \begin{bmatrix} \underset{\sim}{0} \\ -\underset{\sim}{M}^{-1} \end{bmatrix} \underline{\tau} \qquad (19)$$

where

$$\underline{s} = \begin{bmatrix} \underline{\delta\Theta} \\ \underline{\delta\dot{\Theta}} \end{bmatrix} \qquad (20)$$

is the 2N x 1 state vector containing angular and angular rate deviations from nominal. All the submatrices in (19) are N x N. $\underset{\sim}{M}$ represents inertia, $\underset{\sim}{B}$ is a diagonal matrix of damping terms at each joint, $\underset{\sim}{K}$ is a diagonal matrix of spring constants across each joint, and $\underline{\tau}$ is an N vector of the torque sources. A simple and fast computer program returns numerical expressions of the matrices in (19).

Since (19) can easily be shown to be controllable, standard optimal regulator theory can be used to stabilize it about this nominal. Since this requires measurement of both $\underline{\delta\Theta}$ and $\underline{\delta\dot{\Theta}}$, it is useful to know that the system is observable using only measurements of $\underline{\delta\Theta}$.


X'S MARK DPOS 1
◇'S MARK DPOS 2
□'S MARK DPOS 3
■'S MARK DVEL 1
O'S MARK DVEL 2
+'S MARK DVEL 3
DPOS 1
DPOS 2
DPOS 3
DVEL 1
DVEL 2
DVEL 3
TIME (SEC)
DESIRED TRAJECTORY
X'S MARK APOS 1
◇'S MARK APOS 2
□'S MARK APOS 3
■'S MARK AVEL 1
O'S MARK AVEL 2
+'S MARK AVEL 3
APOS 1
APOS 2
APOS 3
AVEL 1
AVEL 2
AVEL 3
TIME (SEC)
TRACKING RESPONSE
THREE LINK ARM TEST NO. 21 SHEET A
X'S MARK TOR 1
◇'S MARK TOR 2
□'S MARK TOR 3
TOR 1
TOR 2
TOR 3
TIME (SEC)
REQUIRED TORQUE
X'S MARK EPOS 1
◇'S MARK EPOS 2
□'S MARK EPOS 3
■'S MARK EVEL 1
O'S MARK EVEL 2
+'S MARK EVEL 3
EPOS 1
EPOS 2
EPOS 3
EVEL 1
EVEL 2
EVEL 3
TIME (SEC)
STATE ERROR
THREE LINK ARM TEST NO. 21 SHEET B

a   b

Fig. 5.

A trajectory control scheme was developed by assuming that the above equations applied to the whole state rather that to deviations from nominal /18/. Optimal linear servo theory /19/ was used to develop feedback gains and control functions. These were then applied to stimulation equations representing all dynamic nonlinearities. A typical result is shown in Figures 5a and 5b, in which angles are in radians and torque in foot-pounds. Although the arm moves at a leisurely pace, the errors are quite small, as are the required torques.

Conclusions

We have presented a unified theory of kinematic rate and position control of computer-driven arms, providing coordination in a variety of coordinate systems. Progress in dynamic control is also described and it is concluded that standard techniques are capable of providing adequate performance when the speed of the arm is not excessive.

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